CHAPTER 4. DETERMINISTIC OPTIMAL CONTROL APPROACH

In this chapter, the optimal control method employed to solve the deterministic formation mission design problem is introduced. First, the formulation of the formation mission design problem is presented. It is formulated as an optimal control problem for a switched dynamical system with logical constraints in disjunctive form. Switched dynamical systems, which are described by both a continuous and a discrete dynamics, are employed to represent the formation flight and logical constraints in disjunctive form are used to model the discrete dynamics of the switched system. In these constraints, binary or integer variables are used to model switching decisions. Finally, the embedding approach is described, which is used to convert the original switched optimal control problem into a smooth optimal control problem, in which the logical constraints in disjunctive form are converted into equality and inequality constraints without binary or integer variables.

4.1. The smooth optimal control problem

Consider a dynamical system described by a differential equation

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \tag{4.1}$$

in which variable t represents the time variable and the functions $x(t)$ and $u(t)$ represent the state and the control input of the system, respectively. Let

$$J\left(t, x(t), u(t)\right), \tag{4.2}$$

be a cost functional, which maps the time variable and the state and control functions into a single number.

An instance of a smooth OCP can be informally stated as follows: determine the control input that steers the system from an initial state to a final state minimizing the cost functional J. In general, the time variable and the control and state functions are subject to constraints.

This simple instance of an OCP corresponds to a single-phase OCP. In general, optimal control models include more than one phase. Each phase in an OCP represents a portion of the trajectory of a dynamical system, and may be subject to different equations of motion, different time discretization, different control parameterizations, or different path constraints.

A different set of ODEs can be employed in each phase to reflect the switching behaviour of a dynamical system. A specific time discretization fitted to the dynamics of that portion of the trajectory can be used in every phase. For instance, if one part of a trajectory has slower dynamics and another part has faster dynamics, the time interval can be split into two phases with coarser time discretization for the first phase and a finer time discretization for the second phase. In the control parameterization method, the control space is discretized by approximating the control function using a linear combination of basis functions. In this way, the OCP is reduced to a NLP with a finite number of decision variables. Different control parameterizations such as piecewise constant and piecewise linear, can be employed in the same problem by defining different phases. Different path constraints, which are bound constraints on some performance parameter, can also be imposed within each phase. Phases are also necessary to impose intermediate constraints upon some variables. This can be done by imposing them as boundary constraints at the junction between two phases. Finally, linkage constraints should be enforced in order to assemble all the phases of the problem to get a unique trajectory for the system [Falck et al., 2021].

OCP can be formulated for continuous-time dynamical systems, as is the case in this thesis, and for discrete-time dynamical systems. OCP formulations may also include binary or integer variables to model decision-making processes.

As mentioned before, in general, optimal control problems include not only differential constraints but also algebraic constraints and path constraints. A more general formulation of a continuous smooth OCP is the following:

$$\min_{u \in ©} \quad J(t, x(t), u(t)) = \mathcal{M}\left(t_F, x_F\right) + \int_{tI}^{tF} \mathcal{L}(t, x(t), u(t)) dt, \tag{4.3a}$$

$$s.t. \quad \dot{x}(t) = f(t, x(t), u(t)), \tag{4.3b}$$

$$0 = g(t, x(t), u(t)), \tag{4.3c}$$

$$\eta_l(t) \leq \eta(t, x(t), u(t)) \leq \eta_u(t), \tag{4.3d}$$

$$x\left(t_I\right) = x_I \in \mathbb{R}^n, \tag{4.3e}$$

$$x\left(t_F\right) = x_F \in \mathbb{R}^n, \tag{4.3f}$$

where $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$, and Eq. (4.3a) represents the performance index of the OCP, in which J is a Bolza cost functional, consisting of two terms, the endpoint cost functional M, also known as Mayer term, which is defined on a neighborhood of B, and the running cost functional, L, also known as Lagrange term, which is a continuously differentiable functional. The control input $u(t)$ is constrained to belong, at each time instant, to the bounded and convex set Ω. Eqs. (4.3b) and (4.3c) represent the set of algebraic-differential equations, where function f determines the dynamics of the system and function g describes the right-hand side of the algebraic constraints. The problem can be also constrained by path constraints represented by Eq. (4.3d), where η_l and η_u represent the lower and upper bound for each function η. Finally, Eqs. (4.3e) and (4.3f) represents the initial and final boundary conditions, respectively. The initial time t_I, the initial state $x(t_I)$, the final time t_F, and final state $x(t_F)$ are assumed to be restricted to a boundary set B as follows: $(t_I, x(t_I), t_F, x(t_F)) \in B = T_I \times B_I \times T_F \times B_F \subset \mathbb{R}^{2n+2}$.

In typical aerospace optimization problems the performance index of Eq. (4.3a) only contains the Mayer term. This is the case of this thesis, in which the objective functional in Eq. (3.18) is in Mayer form and represents a combination of fuel consumption and flight time of the aircraft.

Eqs. (4.3b) and (4.3c) correspond to the DAE systems given in Table 3.2, which describe the motion of each aircraft involved in a formation. In particular, the DAE system in the left column of the table describes the solo flight and the flight of the leader of a formation, and the DAE system in the right column of the table describes the flight of the intermediate and trailing aircraft of a formation.

The path constraints represented by Eq. (4.3d) correspond to the flight envelope given in Eq. (3.4). These constraints are usually given in terms of lower and upper bounds for several state and control variables such as flight altitude, load factor and airspeed. Notice that, Eqs.(4.3c) and (4.3d) also include logical constraints based on the stream-wise distance between aircraft, which are used in this thesis to model the switching between solo flight mode and formation flight mode for each aircraft. These constraints will be discussed later, in Section 4.2.4 of this chapter.

Eqs. (4.3e) and (4.3f) represent the initial and final conditions which are specified for each experiment in chapters 7 and 8.

For the sake of ease of exposition, in the formulation of the OCP for switched dynamical systems presented in the following section, neither algebraic equations nor path constraints have been included.

4.2. The switched optimal control problem

In this thesis, the formation mission design problem is formulated as an OCP for a switched dynamical system. Switched systems are described by both a continuous and a discrete dynamics in which the transitions among discrete states are not established in advance. In particular, each aircraft has different flight modes, namely solo flight and flight in different positions inside a formation, and their combination is represented by the discrete state of the switched dynamical system, which models their joint dynamic behavior. Each flight mode will be represented by different dynamical equations which may include or not formation flight benefits in terms of fuel savings.

In this section, following [Bengea et al., 2011], the formulation of a simple OCP given in Section 4.1, is generalized for switched systems, introducing the switched optimal control problem (SOCP) for two-switched dynamical systems. Hereinafter, subindex "S" has been employed in the formulation of the OCP for switched dynamical systems for the sake of clarity. In particular, the dynamical model of a two-switched dynamical system can be represented by

$$\begin{aligned} \dot{x}_S(t) &= f_{vS(t)}\left(t, x_S(t), u_S(t)\right), \\ x_S(t_I) &= x_I \in \mathbb{R}^n, \\ x_S(t_F) &= x_F \in \mathbb{R}^n, \\ v_S(t) &\in \{0,1\}, t_I \le t \le t_F. \end{aligned} \tag{4.4}$$

In this model, the continuously differentiable vector fields, f_0, f_1 : $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, specify the dynamics of the two possible modes of the system. The control input $u_s(t) \in \Omega \subset \mathbb{R}^m$ is constrained to belong, at each time instant t, to the bounded and convex set Q. The binary variable $\upsilon_s(t)$ *is the mode selection variable that identifies which of the two possible system modes, f_0 or f_1, is active. Thus, both of them, $u_s(t)$ and $\upsilon_s(t)$, can be regarded as control variables. The initial time t_I, the initial state x_s (t_I)*, the final time t_F, and final state $x_s(t_F)$ are assumed to be restricted to a boundary set B as follows: $(t_I, xs(t_I), t_F, xs(t_F)) \in B = T_I \times B_I \times T_f \times B_F \subset \mathbb{R}^{2n+2}$. The performance index of the SOCP has the following form

$$J_S\left(t, x_S(t), u_S(t), v_S(t)\right) = M_S\left(t_F, x_F\right) + \int_{tI}^{tF} L_{vS(t)}\left(t, x_S(t), u_S(t)\right) dt, \tag{4.5}$$

where the endpoint cost function M_s is defined on a neighborhood of B and formally coincide with function M, and L_0 and L_1 are real-valued continuously differentiable functions that represent the running cost of operation of the system in each mode.

The SOCP is thus stated as follows

$$\min_{u_S \in \Omega, v_S \in \{0,1\}} J_S\left(t, x_S(t), u_S(t), v_S(t)\right), \tag{4.6}$$

subject to dynamical equations and endpoint constraints from the set of Eqs. (4.4).

Note that, although the statements of the OCP and the SOCP are quite similar, they are very different in their essence. The fact that the binary variable, $\upsilon_s(t) \in \{0, 1\}$, appears within the SOCP formulation, requires an additional transformation of this kind of problems to enable the use of classical optimal control techniques to solve it. To do so, the embedding approach is introduced in the following section.

4.2.1. Specification of the switched optimal control problem

In the formation mission design problem posed in this thesis, each aircraft has different flight modes, namely solo flight and flight in different positions inside a formation, solo flight (SF) or formation flight (FF), respectively, and their combination is represented by the discrete state of the switched dynamical system, which models their joint dynamic behavior. Therefore, the problem will be formulated as an OCP for a switched dynamical system, following the SOCP formulation. Each flight mode will be represented by different dynamical equations which may include or not formation flight benefits in terms of fuel savings.

4.2.2. The embedding approach

In this section the embedding approach employed to transform a SOCP into an embedded optimal control problem (EOCP) is presented. In this way, a smooth OCP without binary variables is obtained, reducing the computational complexity of finding the solution.

Similarly to previous sections, subindex "E" has been employed in the formulation of the EOCP for sake of clarity. Following again [Bengea et al., 2011], the differential equation of the set of Eqs. (4.4) can be rewritten as follows

$$\dot{x}_E(t) = \left[1 - v_E(t)\right] f_0\left(t, x_E(t), u_{E_0}(t)\right) + v_E(t) f_1\left(t, x_E(t), u_{E_1}(t)\right). \tag{4.7}$$

The dynamical constraint from the set of Eqs. (4.4) and Eq. (4.7) formally coincide under the conditions $\upsilon_E(t) = \upsilon_S(t)$ and $u_{E0}(t) = u_{E1}(t) = u_s(t)$. However, in Eq. (4.7), $\upsilon_E(t) \in [0,1]$, while in Eq. (4.4), $\upsilon_S(t) \in \{0,1\}$, which is the key feature of the embedding approach [Bengea and DeCarlo, 2005, Bengea et al., 2011]. This method is based on solving the SOCP defined above, using only continuous control variables $\upsilon_E(t) \in [0,1]$, $u_{E0}(t)$, $u_{E1}(t) \in \Omega$. In this way, the original switched system is embedded into a larger family of systems parameterized by $\upsilon_E(t) \in [0, 1]$. The performance index of the SOCP from Eq.(4.5) is rewritten as

$$J_E\left(t, x_E(t), u_{E_0}(t), u_{E_1}(t), v_E(t)\right) = \mathcal{M}\left(t_F, x_E\left(t_F\right)\right) \\ + \int_{t_I}^{t_F} \left([1 - v_E(t)]\mathcal{L}_0\left(t, x_E(t), u_{E_0}(t)\right) + v_E(t)\mathcal{L}_1\left(t, x_E(t), u_{E_1}(t)\right)\right)dt. \tag{4.8}$$

The EOCP is thus formulated as

$$\min_{u_{U_0}, u_{E_1} \in \Omega, v_E \in [0,1]} J_E\left(t, x_E(t), u_{E_0}(t), u_{E_1}(t), v_E(t)\right), \tag{4.9}$$

subject to dynamical equations and endpoint constraints from the set of Eqs. (4.7), which is an OCP without binary variables. Therefore, classical techniques from optimal control theory can be applied to solve it.

It has been shown in [Bengea and DeCarlo, 2005] and [Bengea et al., 2011] that, once a solution of the EOCP has been obtained, either the solution is of the switched type, that is, υ_E takes only the values 0 and 1, or suboptimal trajectories of the SOCP can be constructed that can approach the value of the cost for the EOCP arbitrarily closely, and satisfy the boundary conditions within ϵ, with arbitrary $\epsilon > 0$. A thorough discussion about the relationship between the solutions of the SOCP and the EOCP can be found in [Bengea and DeCarlo, 2005] and [Bengea et al., 2011].

In Section 4.2.4, this approach is particularized to the formation mission design problem in which each aircraft can have different dynamic behavior depending on flying in solitaire, or in a different position within the formation.

4.2.3. Logical constraints modeling

The approach proposed in [Wei et al., 2008] has been employed to transform logical constraints in disjunctive form into inequality and equality constraints that involve only continuous auxiliary variables.

It has been shown in [Cavalier et al., 1990] that every Boolean expression can be transformed into conjunctive normal form (CNF). Hence, there is no loss of generality in considering that any logical constraint can be formulated as a CNF expression

$$Q_1 \wedge Q_2 \wedge \ldots \wedge Q_n, \tag{4.10}$$

in conjunctive form, where the symbol $\wedge$ denotes the "and" operator and

$$Q_i = P_i^1 \vee P_i^2 \vee \ldots \vee P_i^{m_i}, \forall i \in \{1, 2, \ldots, n\}, \tag{4.11}$$

where Q_i is expressed in disjunctive form, in which the symbol $\vee$ denotes the "or" operator and the proposition P_i^j is either X_i^j or $\neg X_i^j$. The term X_i^j is a literal that can be either True or False and the symbol $\neg$ represents the "negation" operator.

Term X_i^j represents statements such as "$D_{12}(t) \geq 20b$". Therefore, P_i^j takes the form

$$P_i^j = \left\{ g_i^j \left(x(t) \right) \leq 0 \right\}, \tag{4.12}$$

$\forall i \in \{1, 2, \ldots, n\}$, $\forall j \in \{1, 2, \ldots, m_i\}$, where $g_i^j : \mathbb{R}^{nx} \longrightarrow \mathbb{R}$ is assumed to be a C^1 function.

To incorporate logical constraints in a OCP, they must be converted into a set of equality or inequality constraints. Additionally, if binary variables are not used, the combinatorial complexity of integer programming is avoided. Notice that, conjunctions from Eq. (4.10) can be easily included in a OCP are equivalent to Q_i, $\forall i \in \{1, 2, \ldots, n\}$. To be able to transform the disjunctions into a set of inequality constraints, a continuous variable $\alpha_i^j(t) \in [0, 1]$ is defined and associated with each P_i^j in Eq. (4.12). Thus, Eq. (4.11) can be rewritten as

$$\begin{aligned} \forall i \in \{1, 2, \ldots, n\}, \forall j \in \{1, 2, \ldots, m_i\} : & \alpha_i^j(t) \cdot g_i^j(x(t)) \leq 0, \\ & \text{and } 0 \leq \alpha_i^j(t) \leq 1, \\ & \text{and } \sum_{j=1}^{m_i} \alpha_i^j(t) = 1, \end{aligned} \tag{4.13}$$

The first constraint in (4.13) shows that when $\alpha_i^j(t) = 0$ the constraint $g_i^j(x(t)) \leq 0$ is not forced to be fulfilled. In contrast, if $0 < \alpha_i^j(t) \leq 1$, then $\alpha_i^j(t) \cdot g_i^j\left(x(t)\right) \leq 0$, and, consequently, the constraint $g_i^j(x(t)) \leq 0$ is forced to be fulfilled. The last equation in the system (4.13) guarantees that, at least, one of the propositions P_i^j holds.

In the next section, this approach will be applied to the formation mission design problem in which logical constraints involve the longitudinal distance between aircraft.

4.2.4. Specification of the embedded optimal control problem with equality and inequality constraints

In this section, the embedding approach is applied to the SOCP described in Section 4.2 and the logical constraints in disjunctive form based on the stream-wise distance between aircraft are converted into a set of inequality and equality constraints which will be added to the EOCP.

In the formation mission design problem posed in this thesis, not all the dynamic Eqs. (3.12) are subject to switches. More specifically, the equations of motion associated to the state variables φ, λ, χ, and V, do not switch whereas the aircraft's mass flow rate does when the aircraft p joins or leaves a formation as a trailing or an intermediate aircraft. To model this switching, mass flow rate equations considering fuel savings and not considering them, are combined using the selection variable $\upsilon_{E_p}(t)$, $p \in \{1, \dots, N_a\}$ as follows

$$
\begin{aligned}
&\forall p \in \{1, \dots, N_a\} : \\
&\qquad \dot{\phi}_p = \frac{V_p \cdot \cos\chi_p + V_{W_N}}{R_E + h_p}, \\
&\qquad \dot{\lambda}_p = \frac{V_p \cdot \sin\chi_p + V_{W_E}}{\cos\phi_p \cdot (R_E + h_p)} \\
&\qquad \dot{\chi}_p = \frac{L_p \cdot \sin\mu_p}{V_p \cdot m_p} \\
&\qquad \dot{V}_p = \frac{T_p - D_p}{m_p} \\
&\qquad \dot{m}_p = \left(1 - v_{E_p}\right) \cdot \left[-T_p \cdot \eta_p\right] + v_{E_p} \cdot \left[-\left(1 - \mathcal{R}_{\text{fuel}\,p}\right) \cdot T_p \cdot \eta_p\right]
\end{aligned}
\tag{4.14}
$$

where $\upsilon_{E_p}(t) = 1$ corresponds to FF and $\upsilon_{E_p}(t) = 0$ to SF.

Thus, each aircraft has a different individual dynamic behavior which corresponds to SF or FF in a different position within the in-line formation. The combination of these behaviors represents the discrete states of the switched system that models the joint dynamic behavior of all the aircraft.

However, in mission design problems, the switching control variables $u_{E_p}(t)$, $p \in \{1,\dots,N_a\}$ depend on the state variables, in particular on the geographical coordinates of the aircraft. Indeed, not all transitions between discrete states are possible at any time, since while the distance between aircraft is larger than 20 wingspans, FF can not take place. Furthermore, the switching control variables depend on one another. For instance, in a two-aircraft formation, if one aircraft is selected as the trailing aircraft the other one is forced to be the leader and vice versa. Similarly, in a three-aircraft formation, if one position in the formation is assigned to one aircraft, the other ones must fly in another position.

Thus, the switched system that represents the joint dynamic behavior of all the aircraft involved in the mission design problem should follow some rules that are specified using logical constraints. In particular, the switching logic among the discrete states of the system presented in the SOCP presented in this thesis, are modeled by logical constraints in disjunctive form, based on the stream-wise distance between aircraft. As stated in Eq. (3.7), it has been considered that the great-circle distance between aircraft in FF is between 10 and 20 wingspans, and is greater than 20 wingspans in case of SF. Hence, the distance between aircraft satisfies

$$\begin{aligned} \forall p,q \in \{1,\dots,N_a\}, p<q: \quad & 10b \le \mathcal{D}_{pq}(t) \le 20b, \\ \text{or} \quad & \mathcal{D}_{pq}(t) \ge 20b. \end{aligned} \tag{4.15}$$

Using simple Boolean algebra, Eq.(4.15) can be rewritten as

$$\begin{aligned} \forall p,q \in \{1,\dots,N_a\}, p<q: \quad & 10b \le D_{pq}(t), \\ \text{and} \quad D_{pq}(t) \le 20b \quad & \text{or} \quad D_{pq}(t) \ge 20b. \end{aligned} \tag{4.16}$$

To transform the disjunctions into a set of equality and inequality constraints, a new variable is defined. Thus, in two-aircraft mission design problems, only a continuous variable related to the longitudinal distance D_{12} should be defined, whereas, in three-aircraft mission design problems three continuous variables must be introduced, which are related to the longitudinal distances D_{12}, D_{13}, D_{23}. Thus, calling the new continuous variable $\alpha_{pq}(t) \in [0, 1]$, $\forall\, p, q \in \{1,\dots, N_a\}$, $p < q$, the logical constraints in Eq.(4.16) can be rewritten as

$$\begin{aligned}\forall p,q\in\{1,\dots,N_a\},\ p<q: \quad & \alpha_{pq}(t)\left(D_{pq}(t)-20b\right)\le 0,\\ \text{and}\quad & \left(1-\alpha_{pq}(t)\right)\left(20b-D_{pq}(t)\right)\le 0,\\ \text{and}\quad & 0\le\alpha_{pq}(t)\le 1,\\ \text{and}\quad & D_{pq}(t)\ge 10b.\end{aligned} \tag{4.17}$$

For aircraft p and q, if $D_{pq}(t) > 20b$, $\alpha_{pq}(t)$ must be zero in order to satisfy the second constraint in (4.17). Otherwise, when $D_{pq}(t) \leq 20b$, the variable $\alpha_{pq}(t) = 1$, to fulfill the second constraint in the above set of equations. Formation flight benefits will be achieved only when $\alpha_{pq}(t) = 1$. In any case, the last constraint in (4.17) ensures that the longitudinal distance between aircraft will always satisfy the safety minimum distance of 10b.

It can be observed that nature of the variables $\upsilon_{Ep}(t)$, $\forall p \in \{1, \dots, N_a\}$, and $\alpha_{pq}(t)$, $\forall\, p, q \in \{1, \dots, N_a\}$, $p < q$, is similar. The variable $\upsilon_{Ep}(t)$ selects equations of the dynamic model for aircraft p, whereas the variable $\alpha_{pq}(t)$ selects constraints. Therefore, although, for the sake of clarity, a different notation has been employed in the previous sections to describe them, the variables $\alpha_{pq}(t)$ and $\upsilon_{Ep}(t)$ are directly related by the following equivalence

$$\forall p,q\in\{1,\dots,N_a,\}\ p<q: \quad \alpha_{pq}(t)=v_{E_p}(t), \tag{4.18}$$

where aircraft p is supposed to be the trailing aircraft. Therefore, the variable $\upsilon_{Eq}(t)$, which selects equations of the dynamic model for aircraft q, must be zero regardless of the value of variable $\alpha_{pq}(t)$, as aircraft q will be the leader one and, hence, it will achieve no benefits from the formation.

It is important to notice that the equivalence in Eq. (4.18) between the continuous variables obtained from the embedding approach, used to model the switched dynamical system and transform the logical constraints in disjunctive form into inequality and equality constraints, need not necessarily be fulfilled in others EOCPs.

4.3. Conclusions

In this chapter, the optimal control method employed to solve the deterministic formation mission design problem has been presented. The problem is formulated as an optimal control of a switched dynamical system which has been converted into a smooth optimal control problem using the embedding approach. The embedding approach is a unifying technique able to efficiently tackle both the switching dynamics and the logical constrains of the optimal control problem, transforming it into a smooth optimal control problem, which can be solved using classical optimal control techniques. The main advantages of this approach are that the multiphase formulation is avoided, as well as the use of integer variables, decreasing the computational time and effort in finding the solution. This approach has substantial advantages compared to multiphase approaches, which are based on exhaustively analyzing every possible formation mission individually and then, comparing the results. Additionally, it is easily scalable to design formation mission problems with an arbitrary number of flights.